

Singularity & Regularity Issues for Simplified Models of Turbulence

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Abstract

We consider a family of Leray- α models with periodic boundary conditions in three space dimensions. Such models are a regularization, with respect to a parameter θ , of the Navier-Stokes equations. In particular, they share with the original equation (NS) the property of existence of global weak solutions. We establish an upper bound on the Hausdorff dimension of the time singular set of those weak solutions when θ is subcritical. The result is an interpolation between the bound proved by Scheffer for the Navier-Stokes equations and the regularity result proved in [1].

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1 Introduction

We consider, for $\alpha > 0$ and $0 < \theta < 1/4$, the Leray- α equations in the 3-dimensional flat Torus \mathbb{T}_3

$$(1.1) \quad \begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \mathbb{R}^+ \times \mathbb{T}_3, \\ \bar{\mathbf{u}} = (1 - \alpha^2 \Delta)^{-\theta} \mathbf{u}, & \text{in } \mathbb{T}_3, \\ \nabla \cdot \mathbf{u} = 0, \quad \int_{\mathbb{T}_3} \mathbf{u} = 0, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0. \end{cases}$$

Here the unknowns are the velocity vector field \mathbf{u} and the scalar pressure p . The viscosity ν , the initial velocity vector field \mathbf{u}_0 and the external force \mathbf{f} , with $\nabla \cdot \mathbf{f} = 0$, are given. The nonlocal operator $M_\theta = (1 - \alpha^2 \Delta)^{-\theta}$, acting on $L^2(\mathbb{T}_3, \mathbb{R}^3)$, is defined through the Fourier transform on the torus

$$(1.2) \quad \widehat{M_\theta \mathbf{u}}(\mathbf{k}) = (1 + \alpha^2 |\mathbf{k}|^2)^{-\theta} \widehat{\mathbf{u}}(\mathbf{k}), \quad \mathbf{k} \in \mathbb{Z}^3.$$

The Leray- α equations are among the simplest models of turbulence, introduced nearly a decade ago for numerical simulation purposes. When $\theta = 0$, (1.1) reduces to the Navier Stokes equation for an incompressible fluid. For $\theta > 0$, $\bar{\mathbf{u}}$ is a regularization of the velocity vector field \mathbf{u} . Actually, a crude regularization (or filtering) appeared in the early work of J. Leray [4] where a mollifier was used (i.e., $\bar{\mathbf{u}} = \phi_\varepsilon * \mathbf{u}$) instead of the operator M_θ .

The Leray- α models are approximation of the Navier-Stokes equations and in fact they have several properties in common with (NSE). In particular, (1.1) have existence of weak

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solutions for arbitrary time and large initial-data (see Theorem 2.1).

Our goal, in this short note, is to establish an upper bound for the Hausdorff dimension of the time singular set $\mathcal{S}_\theta(\mathbf{u})$ of weak solutions \mathbf{u} of (1.1). We know, thanks to Scheffer's work [6, 7], that if \mathbf{u} is a weak Leray solution of the Navier-Stokes equations then the $\frac{1}{2}$ -dimensional Hausdorff measure of the time singular set of \mathbf{u} is zero. Further, when $\theta = \frac{1}{4}$, the author in [1] proved the existence of a unique regular weak solution to the Leray- α model (1.1). Therefore, it is interesting to understand how the potential time singular set $\mathcal{S}_\theta(\mathbf{u})$ may depend on the regularization parameter θ .

In fact, we will prove that $\frac{1-4\theta}{2}$ -dimensional Hausdorff measure of the time singular set $\mathcal{S}_\theta(\mathbf{u})$ of any weak solution \mathbf{u} of (1.1) is zero (see Theorem 3.1).

Although we consider only the Leray- α equations (1.1), the same results hold for other models of turbulence as the magnetohydrodynamics MHD- α equations. It was observed in [3] that the qualitative properties of the equation (1.1) relies on the regularization effect of the operator M_θ rather than its explicit form. Indeed, this fact can be checked for the Hausdorff dimension of the time singular set $\mathcal{S}_\theta(\mathbf{u})$.

2 Preliminaries

Before giving some preliminary results we fix some notations. For $p \in [1, \infty)$, the Lebesgue spaces $L^p(\mathbb{T}_3)$, the Sobolev spaces $W^{1,p}(\mathbb{T}_3)$ and the Bochner spaces $L^p(0, T; X)$, $C(0, T; X)$, X being a Banach space, are defined in a standard way. In addition for $s \geq -1$, we introduce the spaces

$$\mathbf{V}^s = \left\{ \mathbf{u} \in W^{s,2}(\mathbb{T}_3)^3, \int_{\mathbb{T}_3} \mathbf{u} = 0, \operatorname{div} \mathbf{u} = 0 \right\},$$

endowed with the norms

$$\|\mathbf{u}\|_{\mathbf{V}^s}^2 = \sum_{\mathbf{k} \in \mathbb{Z}^3} |\mathbf{k}|^{2s} |\widehat{\mathbf{u}}(\mathbf{k})|^2.$$

For the sake of simplicity we introduce the notations

$$\mathbf{H} := \mathbf{V}^0 \quad \text{and} \quad \mathbf{V} := \mathbf{V}^1.$$

2.1 A priori estimates

The essential feature of the operator M_θ is the following regularization effect.

Lemma 2.1 *Let $\theta \in \mathbb{R}_+$, $s \geq -1$ and assume that $\mathbf{u} \in \mathbf{V}^s$. Then $M_\theta \mathbf{u} \in \mathbf{V}^{s+2\theta}$ and*

$$\|M_\theta \mathbf{u}\|_{\mathbf{V}^{s+2\theta}} \leq \frac{1}{\alpha^{2\theta}} \|\mathbf{u}\|_{\mathbf{V}^s}.$$

Next, we prove a priori estimates in the same manner as for the Navier Stokes equations (see [9]). We suppose that \mathbf{u} is a sufficient regular solution of (1.1).

2.1.1 A priori estimates in H

Lemma 2.2 *Let $\mathbf{f} \in L^2([0, T], \mathbf{V}^{-1})$ and $\mathbf{u}_0 \in \mathbf{H}$, for all $T \geq 0$ there exists $K_1(T)$ and $K_2(T)$ such that any solution \mathbf{u} of (1.1) satisfies*

$$(2.1) \quad \|\mathbf{u}\|_{L^2([0, T], \mathbf{V})}^2 \leq K_1(T), \quad \text{where } K_1(T) = \frac{1}{\nu} \left(\|\mathbf{u}_0\|_{\mathbf{H}}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{L^2([0, T], \mathbf{V}^{-1})}^2 \right)$$

and

$$(2.2) \quad \|\mathbf{u}\|_{L^\infty([0, T], \mathbf{H})}^2 \leq K_2(T), \quad \text{where } K_2(T) = \nu K_1(T).$$

Proof Taking the L^2 -inner product of the first equation of (1.1) with \mathbf{u} and integrating by parts. Using the incompressibility of the velocity field and the duality relation we obtain

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|_{\mathbf{H}}^2 + \nu \|\nabla \mathbf{u}\|_{\mathbf{H}}^2 = \int_{\mathbb{T}_3} \mathbf{f} \cdot \mathbf{u} dx \leq \|\mathbf{f}\|_{\mathbf{V}^{-1}} \|\mathbf{u}\|_{\mathbf{V}}.$$

Using Young inequality we get

$$\frac{d}{dt} \|\mathbf{u}\|_{\mathbf{H}}^2 + \nu \|\nabla \mathbf{u}\|_{\mathbf{H}}^2 \leq \frac{1}{\nu} \|\mathbf{f}\|_{\mathbf{V}^{-1}}^2.$$

Integration with respect to time gives the desired estimates. ■

2.1.2 A priori estimates in V

Now we use the regularization effect of Lemma 2.1 to prove the following a priori estimates.

Lemma 2.3 *Let $\mathbf{f} \in L^2([0, T], \mathbf{H})$ and $\mathbf{u}_0 \in \mathbf{V}$. Assume that $0 < \theta < 1/4$. Then there exists $T_* := T_*(\mathbf{u}_0)$ and $M(T_*) < \infty$ such that the solution \mathbf{u} of (1.1) satisfies*

$$\sup_{t \in [0, T_*]} \|\mathbf{u}\|_{\mathbf{V}}^2 \leq 2(1 + \|\mathbf{u}_0\|_{\mathbf{V}}^2)$$

and

$$\int_0^{T_*} \|\Delta \mathbf{u}(t)\|_{\mathbf{H}}^2 dt \leq M(T_*).$$

Proof Taking the L^2 -inner product of the first equation of (1.1) with $-\Delta \mathbf{u}$ and integrating by parts. Using the incompressibility of the velocity field and the duality relation combined with Hölder inequality and Sobolev injection, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}\|_{\mathbf{H}}^2 + \nu \|\Delta \mathbf{u}\|_{\mathbf{H}}^2 &\leq \int_{\mathbb{T}_3} |(\bar{\mathbf{u}} \cdot \nabla) \mathbf{u} \Delta \mathbf{u}| dx + \int_{\mathbb{T}_3} |\mathbf{f} \Delta \mathbf{u}| dx \\ &\leq \alpha^{-2\theta} \|\nabla \mathbf{u}\|_{\mathbf{H}} \|\nabla \mathbf{u}\|_{\mathbf{V}^{\frac{1}{2}-2\theta}} \|\Delta \mathbf{u}\|_{\mathbf{H}} + \|\mathbf{f}\|_{\mathbf{H}} \|\Delta \mathbf{u}\|_{\mathbf{H}}. \end{aligned}$$

Interpolating between \mathbf{V}^1 and \mathbf{V}^2 we get

$$\frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}\|_{\mathbf{H}}^2 + \nu \|\Delta \mathbf{u}\|_{\mathbf{H}}^2 \leq \alpha^{-2\theta} \|\nabla \mathbf{u}\|_{\mathbf{H}}^{\frac{3}{2}+2\theta} \|\Delta \mathbf{u}\|_{\mathbf{H}}^{\frac{3}{2}-2\theta} + \|\mathbf{f}\|_{\mathbf{H}} \|\Delta \mathbf{u}\|_{\mathbf{H}}.$$

Using Young inequality we get

$$(2.3) \quad \frac{d}{dt} \|\nabla \mathbf{u}\|_{\mathbf{H}}^2 + \nu \|\Delta \mathbf{u}\|_{\mathbf{H}}^2 \leq \frac{1}{\nu} \|\mathbf{f}\|_{\mathbf{H}}^2 + C(\alpha, \theta) \|\nabla \mathbf{u}\|_{\mathbf{H}}^{\frac{2(3+4\theta)}{1+4\theta}}.$$

We get a differential inequality

$$(2.4) \quad Y' \leq C(\alpha, \theta, \nu, f) Y^\gamma,$$

where

$$Y(t) = 1 + \|\mathbf{u}\|_{\mathbf{V}}^2 \quad \text{and} \quad \gamma = \frac{3 + 4\theta}{1 + 4\theta}.$$

We conclude that

$$Y(t) \leq \frac{Y(0)}{(1 - 2Y(0)^{\gamma-1} C(\alpha, \theta, \nu, f) t)^{\frac{1}{\gamma-1}}}$$

as long as $t < \frac{1}{2Y(0)^{\gamma-1} C(\alpha, \theta, \nu, f)}$. Hence we obtain

$$\sup_{t \in [0, T_*]} \|\mathbf{u}\|_{\mathbf{V}}^2 \leq 2(1 + \|\mathbf{u}_0\|_{\mathbf{H}}^2)$$

$$(2.5) \quad \text{with } T_* := \frac{3}{8C(\alpha, \theta, \nu, f)} \frac{1}{(1 + \|\mathbf{u}_0\|_{\mathbf{V}}^2)^{\gamma-1}}.$$

Integrating (2.3) with respect to time on $[0, T_*]$ gives the desired estimates

$$\int_0^{T_*} \|\Delta \mathbf{u}(t)\|_{\mathbf{H}}^2 dt \leq M(T_*),$$

where

$$M(T_*) = \frac{1}{\nu} \left(\|\mathbf{u}_0\|_{\mathbf{H}}^2 + \frac{2}{\nu} \int_0^{T_*} \|\mathbf{f}\|_{\mathbf{H}}^2 dt + C(\alpha, \theta) [2(1 + \|\mathbf{u}_0\|_{\mathbf{H}}^2)]^\gamma \right).$$

■

2.2 Existence and uniqueness results

The next two theorems collect the most typical results for the Leray- α models of turbulence (see [1], [5]). The proofs of these two theorems follow by combination of the above a priori estimates with a Galerkin method. This is a classical argument which we avoid its repetition. For further information, we refer the reader to [9], [1] and the references therein.

By $C_{weak}([0, T]; \mathbf{H})$ we denote the vector space of all mappings $\mathbf{v} : [0, T] \rightarrow \mathbf{H}$ such that for any $\mathbf{h} \in \mathbf{H}$, the function

$$t \mapsto \int_{\mathbb{T}_3} \mathbf{v}(t) \mathbf{h} d\mathbf{x}$$

is continuous on $[0, T]$.

Theorem 2.1 *Let $\mathbf{f} \in L^2([0, T], \mathbf{V}^{-1})$ and $\mathbf{u}_0 \in \mathbf{H}$. Assume that $0 \leq \theta < 1/4$. Then for any $T > 0$ there exist a weak Leray solution $(\mathbf{u}, p) := (\mathbf{u}_\alpha, p_\alpha)$ to (1.1) such that*

$$\mathbf{u} \in C_{weak}([0, T]; \mathbf{H}) \cap L^2([0, T]; \mathbf{V}), \quad \frac{\partial \mathbf{u}}{\partial t} \in L^{\frac{5}{3-2\theta}}([0, T]; W^{-1, \frac{5}{3-2\theta}}(\mathbb{T}_3)^3),$$

$$p \in L^{\frac{5}{3-2\theta}}([0, T], L^{\frac{5}{3-2\theta}}(\mathbb{T}_3)),$$

$$\begin{aligned}
(2.6) \quad & \int_0^T \left\langle \frac{\partial \mathbf{u}}{\partial t}, \mathbf{w} \right\rangle - (\bar{\mathbf{u}} \otimes \mathbf{u}, \nabla \mathbf{w}) + \nu(\nabla \mathbf{v}, \nabla \mathbf{w}) - (p, \operatorname{div} \mathbf{w}) \, dt \\
& = \int_0^T \langle \mathbf{f}, \mathbf{w} \rangle \, dt \quad \text{for all } \mathbf{w} \in L^{\frac{5}{2+2\theta}}(0, T; W^{1, \frac{5}{2+2\theta}}(\mathbb{T}_3)^3),
\end{aligned}$$

where the velocity \mathbf{u} verifies

$$(2.7) \quad \sup_{t \in (0, T)} \|\mathbf{u}(t)\|_{\mathbf{H}}^2 + \int_0^T \|\mathbf{u}(t)\|_{\mathbf{V}}^2 \, dt \leq \|\mathbf{u}_0\|_{\mathbf{H}}^2 + \int_0^T \langle \mathbf{f}, \mathbf{u} \rangle \, dt,$$

and the initial data is attained in the following sense

$$\lim_{t \rightarrow 0+} \|\mathbf{u}(t) - \mathbf{u}_0\|_{\mathbf{H}}^2 = 0.$$

Remark 2.1 If $\theta = \frac{1}{4}$ a weak solution to Leray- α model is called regular weak solution [1], in addition, the solution is unique and it satisfies

$$\mathbf{u} \in C([0, T]; \mathbf{H}) \cap L^2([0, T]; \mathbf{V}), \quad \frac{\partial \mathbf{u}}{\partial t} \in L^2([0, T]; \mathbf{V}^{-1}), \quad \text{and} \quad p \in L^2([0, T], L^2(\mathbb{T}_3)).$$

In this case one also has energy equality in (2.7) instead of inequality.

Theorem 2.2 Let $\mathbf{f} \in L^2([0, T], \mathbf{H})$ and $\mathbf{u}_0 \in \mathbf{V}$. Assume that $0 \leq \theta < 1/4$. Then there exists $T_* := T_*(\mathbf{u}_0)$, determined by (2.5), and there exists a unique strong solution \mathbf{u} to (1.1) on $[0, T_*]$ satisfying:

$$\begin{aligned}
& \mathbf{u} \in C([0, T_*]; \mathbf{V}) \cap L^2([0, T_*]; \mathbf{V}^2), \\
& \frac{\partial \mathbf{u}}{\partial t} \in L^2([0, T_*]; L^2(\mathbb{T}_3)^3) \quad \text{and} \quad p \in L^2([0, T_*], W^{1,2}(\mathbb{T}_3)).
\end{aligned}$$

Remark 2.2 If $\theta = \frac{1}{4}$ the strong solution to the Leray- α model exists for any arbitrary time $T > 0$. Indeed, when $\theta = \frac{1}{4}$, $\gamma = \frac{3+4\theta}{1+4\theta} = 2$, the differential inequality (2.4) becomes

$$(2.8) \quad Y' \leq C(\alpha, \theta, \nu, f) Y^2.$$

Thus, using Gronwall's inequality ($Y \in L^1[0, T]$) we obtain the desired result.

Remark 2.3 Let us assume that $\mathbf{f} \in L^2([0, T], \mathbf{H})$, $\mathbf{u}_0 \in \mathbf{V}$ and $0 < \theta < 1/4$. With the local existence of strong solution and the weak=strong theorem of Serrin [8], the solution \mathbf{u} is automatically strong and thus smooth on $[0, T_*) \times \mathbb{T}_3$, where $T_* \in [0, T]$. We note that in Theorem 2.1, a weak solution satisfying the above properties with $T = \infty$ exists for every divergence-free $\mathbf{u}_0 \in \mathbf{H}$.

3 The Main Result And Its Proof

The basic facts about Hausdorff measures can be found for instance in [2]. We recall here the definition of those measures.

Definition 3.1 Let X be a metric space and let $a > 0$. The a -dimensional Hausdorff measure of a subset Y of X is

$$\mu_a(Y) = \lim_{\epsilon \searrow 0} \mu_{a,\epsilon}(Y) = \sup_{\epsilon > 0} \mu_{a,\epsilon}(Y)$$

where

$$\mu_{a,\epsilon}(Y) = \inf \sum_j (\text{diameter } B_j)^a,$$

the infimum being taken over all the coverings of Y by balls B_j such that $\text{diameter } B_j \leq \epsilon$.

Definition 3.2 Let $T > 0$. We define the time singular set $\mathcal{S}_\theta(\mathbf{u})$ of $\mathbf{u}(t)$, a weak solution of (1.1) given by Theorem 2.1, as the set of $t \in [0, T]$ such that $\mathbf{u}(t) \notin \mathbf{V}$.

The main result of the paper is the following theorem.

Theorem 3.1 Let \mathbf{u} be any weak Leray solution to eqs. (1.1) given by Theorem 2.1 (We suppose that the external force $\mathbf{f} \in L^2([0, T], \mathbf{H})$). Then for any $T > 0$ the $\frac{1-4\theta}{2}$ -dimensional Hausdorff measure of the time singular set $\mathcal{S}_\theta(\mathbf{u})$ of \mathbf{u} is zero.

The rest of the paper is devoted to the proof of the main Theorem. The following Lemma characterizes the structure of the time singularity set of a weak solution of (1.1).

Lemma 3.1 We assume that $\mathbf{u}_0 \in \mathbf{H}$, $\mathbf{f} \in L^2([0, T], \mathbf{H})$ and \mathbf{u} is any weak solution of (1.1) given by Theorem 2.1. Then there exist an open set \mathcal{O} of $(0, T)$ such that:

- (i) For all $t \in \mathcal{O}$ there exist $t \in (t_1, t_2) \subseteq (0, T)$ such that $\mathbf{u} \in C((t_1, t_2), \mathbf{V})$.
- (ii) The Lebesgue measure of $[0, T] \setminus \mathcal{O}$ is zero.

Proof. Since $\mathbf{u} \in C_{weak}([0, T]; \mathbf{H})$, $\mathbf{u}(t)$ is well defined for every t and we can define

$$\Sigma = \{t \in [0, T], \mathbf{u}(t) \in \mathbf{V}\},$$

$$\Sigma^c = \{t \in [0, T], \mathbf{u}(t) \notin \mathbf{V}\},$$

$$\mathcal{O} = \{t \in (0, T), \exists \epsilon > 0, \mathbf{u} \in C((t - \epsilon, t + \epsilon), \mathbf{V})\}.$$

It is clear that \mathcal{O} is open. Since $\mathbf{u} \in L^2([0, T]; \mathbf{V})$, Σ^c has Lebesgue measure zero. Let us take t_0 such that $t_0 \in \Sigma$, and $t_0 \notin \mathcal{O}$, then according to Theorem 2.2, there exists $\epsilon > 0$ such that $\mathbf{u} \in C((t_0, t_0 + \epsilon), \mathbf{V})$. So that, t_0 is the left end of one of the connected components of \mathcal{O} . Thus $\Sigma \setminus \mathcal{O}$ is countable and consequently $[0, T] \setminus \mathcal{O}$ has Lebesgue measure zero. This finishes the proof. \blacksquare

Remark 3.1 We deduce from Theorem 2.2 that, if (α_i, β_i) , $i \in I$, is one of the connected components of \mathcal{O} , then

$$\lim_{t \rightarrow \beta_i} \|\mathbf{u}(t)\|_{\mathbf{V}} = +\infty.$$

Indeed, otherwise Theorem 2.2 would show that there exist an $\epsilon > 0$ such that $\mathbf{u} \in C((\beta_i, \beta_i + \epsilon), \mathbf{V})$ and β_i would not be the end point of a connected component of \mathcal{O} .

Lemma 3.2 Under the same notations of Lemma 3.1. Let (α_i, β_i) , $i \in I$, be the connected components of \mathcal{O} . Then

$$(3.1) \quad \sum_{i \in I} (\beta_i - \alpha_i)^{\frac{1-4\theta}{2}} < \infty$$

Proof. Let (α_i, β_i) be one of these connected components and let $t \in (\alpha_i, \beta_i) \subseteq \mathcal{O}$. Since $\mathbf{u} \in C_{weak}([0, T]; \mathbf{H}) \cap L^2([0, T]; \mathbf{V})$, $\mathbf{u}(t)$ is well defined for every $t \in (\alpha_i, \beta_i)$ and t can be chosen such that $\mathbf{u}(t) \in \mathbf{V}$. According to Theorem 2.2, inequality (2.5) and the fact that $\|\mathbf{u}(\beta_i)\|_{\mathbf{V}} = +\infty$, we have for $t \in (\alpha_i, \beta_i)$

$$\beta_i - t \geq \frac{1}{C(\alpha, \theta, \nu, f)} \frac{1}{(1 + \|\mathbf{u}(t)\|_{\mathbf{V}}^2)^{\gamma-1}},$$

where we have used that $\gamma = \frac{3+4\theta}{1+4\theta} > 1$. Thus

$$\frac{C(\alpha, \theta, \nu, f)}{(\beta_i - t)^{\frac{1}{\gamma-1}}} \leq 1 + \|\mathbf{u}(t)\|_{\mathbf{V}}^2.$$

Then we integrate on (α_i, β_i) to obtain

$$C(\alpha, \theta, \nu, f)(\beta_i - \alpha_i)^{\frac{-1}{\gamma-1}+1} \leq (\beta_i - \alpha_i) + \int_{\alpha_i}^{\beta_i} \|\mathbf{u}(t)\|_{\mathbf{V}}^2 dt,$$

Adding all these relations for $i \in I$ we obtain

$$C(\alpha, \theta, \nu, f) \sum_{i \in I} (\beta_i - \alpha_i)^{\frac{-1}{\gamma-1}+1} \leq T + \int_0^T \|\mathbf{u}(t)\|_{\mathbf{V}}^2 dt.$$

■

Proof of Theorem 3.1. We set $\mathcal{S} = \mathcal{S}_\theta(\mathbf{u}) = [0, T] \setminus \mathcal{O}$. We have to prove that the $\frac{1-4\theta}{2}$ -dimensional Hausdorff measure of \mathcal{S} is zero. Since the Lebesgue measure of \mathcal{O} is finite ,i.e.

$$(3.2) \quad \sum_{i \in I} (\beta_i - \alpha_i) < \infty,$$

it follows from Lemma 3.2 that for every $\epsilon > 0$ there exist a finite part $I_\epsilon \subset I$ such that

$$(3.3) \quad \sum_{i \in I \setminus I_\epsilon} (\beta_i - \alpha_i) \leq \epsilon$$

and

$$(3.4) \quad \sum_{i \in I \setminus I_\epsilon} (\beta_i - \alpha_i)^{\frac{1-4\theta}{2}} \leq \epsilon$$

Note that $\mathcal{S} \subset [0, T] \setminus \bigcup_{i \in I_\epsilon} (\alpha_i, \beta_i)$ and the set $[0, T] \setminus \bigcup_{i \in I_\epsilon} (\alpha_i, \beta_i)$ is the union of finite number of mutually disjoint closed intervals, say B_j , for $j = 1, \dots, N$. Our aim now is to show that the diameter $B_j \leq \epsilon$. Since the intervals (α_i, β_i) are mutually disjoint, each interval (α_i, β_i) , $i \in I \setminus I_\epsilon$, is included in one, and only one, interval B_j . We denote by I_j the set of indice i such that $(\alpha_i, \beta_i) \subset B_j$. It is clear that $I_\epsilon, I_1, \dots, I_N$ is a partition of I and we have $B_j = (\bigcup_{i \in I_j} (\alpha_i, \beta_i)) \cup (B_j \cap \mathcal{S})$ for all $j = 1, \dots, N$. It follows from (3.2) that

$$(3.5) \quad \text{diameter } B_j = \sum_{i \in I_j} (\beta_i - \alpha_i) \leq \epsilon.$$

Finally in virtue of the definition 3.1 and estimates (3.5), (3.4) and since $l^\delta \hookrightarrow l^1$ for all $0 < \delta < 1$ we have

$$\begin{aligned}
(3.6) \quad \mu_{\frac{1-4\theta}{2}, \epsilon}(\mathcal{S}) &\leq \sum_{j=1}^N (\text{diameter } B_j)^{\frac{1-4\theta}{2}} \\
&\leq \sum_{j=1}^N \left(\sum_{i \in I_j} (\beta_i - \alpha_i) \right)^{\frac{1-4\theta}{2}} \\
&\leq \sum_{j=1}^N \sum_{i \in I_j} (\beta_i - \alpha_i)^{\frac{1-4\theta}{2}} \\
&= \sum_{i \in I \setminus I_\epsilon} (\beta_i - \alpha_i)^{\frac{1-4\theta}{2}} \leq \epsilon.
\end{aligned}$$

Letting $\epsilon \rightarrow 0$, we find $\mu_{\frac{1-4\theta}{2}}(\mathcal{S}) = 0$ and this completes the proof. ■

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